An application to Ramsey theory

Given two graphs G, H, the **Ramsey number** $R(G,H)$ is the minimum value N such that in any 2-edge-coloring of the complete graph K_N there is a monochromatic copy of G or H. Of particular interest is the case when $G = H = K_n$. Note that when working with 2 colors, we can have one color specify edges and the other color specify non-edges, so we would like to determine N such that every graph on N vertices contains a clique or independent set of size n.

Recall the following bounds on $R(K_n, K_n)$ for $n \geq 4$.

$$2^{n/2} \leq R(K_n, K_n) \leq 2^{2n}.$$

That is, any 2-edge-coloring of the complete graph on 4^n vertices contains a monochromatic K_n.

We will need to following strengthening of the earlier embedding lemma.

Lemma 1 (Improved embedding lemma)**.** *Let F be a k-chromatic graph with maximum degree $\Delta(F)$. Fix $0 < \delta < \frac{1}{k}$ and let G be a graph and let $V_1, \dots, V_k$ be disjoint sets of vertices of G. If each V_i has $|V_i| \geq 2\delta^{-\Delta}|F|$ and each pair of partition classes is $\frac{1}{2\Delta}\delta^{\Delta}$-regular with density $\geq 2\delta$, then the classes $V_1, V_2, \dots, V_k$ contain F.*

Proof. □

Surprisingly, if we choose G and H to be graphs with bounded maximum degree (compared to n), then the order of $R(G,H)$ decreases dramatically.

Theorem 2 (Chvátal-Rödl-Szemerédi-Trotter, 1983)**.** *For $\Delta > 0$, there exists $c = c(\Delta)$ such that if G and H are n-vertex graphs of maximum degree at most Δ and n large enough, then*

$$R(G,H) \leq cn.$$

That is, any 2-edge-coloring of K_{cn} contains a monochromatic G or H.

Proof of Theorem 2. **1:** Here is outline of the proof, try fill in the details. Fill in the constants.

Take any 2-edge-coloring (red/blue) of K_{cn}. Goal is to find monochromatic G or H. Apply regularity lemma on the red graph. Make an $(\epsilon, 0)$-reduced graph R. Show using Turán's theorem, that R contains K_m (you will need to pick m). Color an edge of K_m green, if density of (red) edges between the corresponding clusters is $\geq \frac{1}{2}$ and color by yellow otherwise. Use Ramsey theorem, to find a monochromatic $K_{\Delta+1}$. The use embedding lemma to find monochromatic G or H. See next exercise for hint what to do if $K_{\Delta+1}$ is yellow.

Solution: Recall, that for any graph G that $\chi(G) \leq \Delta(G) + 1$, so put $0 < \delta < \frac{1}{\Delta+1}$. Put $\epsilon = \frac{1}{2\Delta}\delta^{\Delta}$ and $m = 4^{\Delta+1}$ and let $M = M(\epsilon, 2m)$ be as in the regularity lemma. Now put $c = 2\delta^{-\Delta}M$ and consider an arbitrary 2-edge-coloring with blue and red of K_{cn}.

Now apply the regularity lemma to the graph induced by the red edges with parameters ϵ and $2m$ defined above and let $V_1, \dots V_r$ be the Szemerédi partition with $2m < r < M(\epsilon, 2m)$. Observe that

$$|V_i| \geq \frac{cn}{r} \geq \frac{cn}{M} = 2\delta^{-\Delta}n.$$

Consider the graph R with the clusters $V_1, \dots, V_r$ as vertices and two clusters form an edge if they are ϵ-regular (this is actually the $(\epsilon, 0)$-reduced graph). The number of

ϵ-regular pairs in is the number of edges in R, so we have at most ϵr^2 of the pairs of clusters are not ϵ-regular, so

$$e(R) \geq \binom{r}{2} - \epsilon r^2 = \left(1 - \frac{1}{r} - 2\epsilon\right)\frac{r^2}{2} > \left(1 - \frac{1}{m-1}\right)\frac{r^2}{2}.$$

Thus, by Turán's theorem, R contains a K_m. That is, there is a collection of clusters $V_1, \ldots, V_m$ that are all pairwise ϵ-regular.

Now color the edges of the K_m as follows. If the corresponding clusters V_i, V_j have density $\geq \frac{1}{2}$, then color green. Otherwise, if V_i, V_j has density $< \frac{1}{2}$, then color yellow. This gives a 2-edge-coloring of K_m. Now, because $m = 4^{\Delta+1}$, by the version of Ramsey's theorem above gives that K_m contains a monochromatic $K_{\Delta+1}$. That is, R contains a monochromatic $K_{\Delta+1}$. Suppose first that the $K_{\Delta+1}$ is green. Then this complete graph corresponds to clusters $V_1, \ldots, V_{\Delta+1}$ such that all pairs of clusters are ϵ-regular and have density $\geq \frac{1}{2}$. Therefore, we may apply the embedding lemma to find a red copy of G.

Suppose otherwise that the monochromatic $K_{\Delta+1}$ is yellow. This implies that the clusters $V_1, \ldots, V_{\Delta+1}$ are such that all pairs have density $< \frac{1}{2}$ among red edges, thus density $\geq \frac{1}{2}$ among blue edges. Furthermore, because the pairs are ϵ-regular among red edges, they are ϵ-regular among blue edges (Exercise). Therefore, we may apply the embedding lemma to find a blue copy of H.

$\square$

2: Show that if a bipartite graph is ϵ-regular, then the complement is also ϵ-regular.

Hint: What is the relationship between the density of two sets in the graph and its complement?

Solution: Suppose A and B are a pair of sets with density $d(A, B)$ and let $\overline{d}(A, B)$ denote their density in the complement graph. Clearly,

$$d(A, B) + \overline{d}(A, B) = \frac{e(A, B)}{|A||B|} + \frac{|A||B| - e(A, B)}{|A||B|} = 1.$$

Thus

$$\overline{d}(A, B) = 1 - d(A, B).$$

Suppose A and B are the classes of a bipartite graph and the pair is ϵ-regular. Consider subsets $A' \subseteq A$ and $B' \subseteq B$ that satisfy $|A'| \geq \epsilon|A|$ and $|B'| \geq \epsilon|B|$. Considering the complement, we have

$$|\overline{d}(A, B) - \overline{d}(A', B') = |1 - d(A, B) - (1 - d(A', B'))| = |d(A, B) - d(A', B'))| < \epsilon$$

where the last inequality is by the ϵ-regularity of the pair A, B.